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Deformations of Lie algebras of vector fields arising from families of schemes

Friedrich Wagemann*

Université de Nantes, France

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Abstract

Fialowski and Schlichenmaier constructed global deformations of the infinitesimally and formally rigid Lie algebra of polynomial vector fields on the circle from families of projective curves with marked points. In the present article, we show how to obtain these examples in a conceptual way. For this, we define a stack of deformations of Lie algebras and study the morphism from the moduli stack to this deformation stack which associates to a family of marked curves the Lie algebra of vertical vector fields on the punctured family.

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0. Introduction

The goal of the present paper is to construct examples of global deformations of vector field Lie algebras in a conceptual way. Fialowski and Schlichermaier [8] constructed global deformations of the infinitesimally and formally rigid Lie algebra of polynomial vector fields on the circle and of its central extension, the Virasoro algebra. The Lie algebra of polynomial vector fields on the circle is replaced by/seen here as the Lie algebra of meromorphic or rational vector fields on the Riemann sphere admitting poles only in the points 0 and ∞. In this context one gets non-trivial deformations from an affine family of curves by first deforming it into a projective family with marked points and then extracting the points. Fialowski and Schlichermaier obtain in this way non-trivial deformations of Lie algebras, and the underlying families of curves present singularities.

In the attempt to produce deformations of vector field Lie algebras from deformations of the underlying pointed algebraic variety in a general framework, we are led to a notion of *global deformation* which is different from the one used by Fialowski and Schlichenmaier, one which is closer to deformations in algebraic geometry. The first goal is to compare these global deformations with the corresponding notion from Fialowski–Schlichermaier's article.

* Tel.: +33 2 51 12 59 57.
E-mail address: wagemann@math.univ-nantes.fr.

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We then stick to the notion of deformations of the Lie algebra of vector fields on a pointed algebraic curve imposed by the deformation of the underlying curve, and show its close relation to the moduli space of pointed curves. In order to formulate this relation, we show that the "space of deformations" carries the geometric structure of a $\mathbb{C}$-stack. This is a kind of functor from the category of model spaces, here affine schemes, to the category of groupoids (in order to capture the idea of considering deformations up to isomorphism). It has to satisfy conditions to provide construction of geometric objects by gluing local data, making it resemble the functor of points of a scheme. The link between the moduli stack $\mathcal{M}_{g,n}$ and the deformation stack $\mathcal{D}\mathrm{ef}$ is a morphism of stacks I. It realizes a family of marked projective curves as a deformation of the Lie algebra of regular vector fields on the affine curve obtained from extracting the points, by extracting the divisor associated to the union of the marked points, and then taking the Lie algebra of sections of the relative tangent bundles of the family.

We show furthermore that I is almost a monomorphism. The idea that this might be true comes from Pursell–Shanks theory of describing the underlying manifold (and more geometric objects related to it) by its Lie algebra of tangent vector fields. We generalize this theory to a relative (affine) setting, i.e. we show that over $\mathbb{C}$, $\mathrm{Der}_A(B) \cong \mathrm{Der}_A(B')$ implies an A-isomorphism $B \cong B'$.

Let us observe that all which is discussed in this paper can be easily transposed to current algebras, instead of vector field Lie algebras. We hope that the stack approach to deformations is useful in the given situation: our belief is that it may give rise to a cohomology theory governing global deformations by associating some standard homological algebra tool to the above morphism of stacks I (its cotangent complex ?). For this direction, it might be necessary to show first that some substack of our deformation stack is an algebraic stack, cf the remark at the end of Section 5.

1. Global deformations of general and product types

Let $\mathbb{C}$ be the base field; we are conscious that some results are valid in a much more general framework.

In this first section, we compare two types of global deformations of Lie algebras. On the one hand, we have global deformations as Fialowski–Schlichenmaier [8] defined them. We will call these of *product type*. In the following, A is an associative commutative unital $\mathbb{C}$-algebra (and all $\mathbb{C}$-algebras will be supposed associative commutative and unital).

Definition 1. For a Lie algebra $\mathfrak{g}_0$, a global deformation (of product type) with base A is an A-Lie algebra structure $[-,-]_\lambda$ on the tensor product $\mathfrak{g}_0 \otimes_{\mathbb{C}} A$, together with a morphism of Lie algebras $\mathrm{id} \otimes \epsilon : \mathfrak{g}_0 \otimes A \to \mathfrak{g}_0 \otimes \mathbb{C} \simeq \mathfrak{g}_0$, where ϵ is the augmentation of the (augmented) algebra A.

Concretely, the bracket $[-,-]_\lambda$ is antisymmetric, satisfies the Jacobi identity, and the following two identities for $a, b \in A$ and $x, y \in \mathfrak{g}_0$ owing to A-linearity of the bracket and the fact that $\epsilon \otimes \mathrm{id}$ is supposed to be a Lie algebra homomorphism:

(1) $[x \otimes a, y \otimes b]_\lambda = [x \otimes 1, y \otimes 1]_\lambda (ab)$
(2) $(\mathrm{id} \otimes \epsilon)([x \otimes 1, y \otimes 1]_\lambda) = [x, y] \otimes 1.$

By condition (1), it is enough to give the commutators $[x \otimes 1, y \otimes 1]_\lambda$ for all $x, y \in \mathfrak{g}_0$ in order to describe the deformation. By condition (2), their bracket has the form

$$[x \otimes 1, y \otimes 1]_\lambda = [x \otimes 1, y \otimes 1] + \sum_i z_i \otimes a_i.$$

We call the notion with which we want to contrast these global deformations of product type *general global deformations*. We take over the term "global" from an earlier work on the subject, notwithstanding the fact that these deformations are "only" on affine open sets (namely, on the affine open set given by $\mathrm{Spec}(A)$, the spectrum of A).

Definition 2. Let $\mathfrak{g}_0$ be a $\mathbb{C}$-Lie algebra. A general global deformation of $\mathfrak{g}_0$ is an A-Lie algebra $\mathfrak{g}$ together with an augmentation $\epsilon : A \to \mathbb{C}$ such that $\mathfrak{g} \otimes_A \mathbb{C} \simeq \mathfrak{g}_0$ as $\mathbb{C}$-Lie algebras (where A acts on $\mathbb{C}$ via ϵ).

One detail is hidden in this definition. $\mathfrak{g} \otimes_A \mathbb{C}$ is a $\mathbb{C}$-Lie algebra: indeed, $\mathfrak{g} \otimes_A \mathbb{C}$ is a $\mathbb{C}$-Lie algebra with the current algebra bracket $[x \otimes \alpha, y \otimes \beta] := [x, y] \otimes \alpha\beta$ which we call the *trivial bracket* here. Then the space

$$I := \mathrm{span}_{\mathbb{C}}\{ax \otimes \alpha - x \otimes \epsilon(a)\alpha \mid x \in \mathfrak{g}_0, a \in A, \alpha \in \mathbb{C}\}$$

is a Lie-ideal as it follows directly from the A-linearity of the bracket on $\mathfrak{g}$, so the quotient $\mathfrak{g} \otimes_A \mathbb{C}$ is a Lie algebra.

Proposition 1. *A deformation of product type* $(\mathfrak{g}_0 \otimes_{\mathbb{C}} A, [,]_\lambda, \epsilon)$ *is a general global deformation.*

Proof. It is clear that $(\mathfrak{g}_0 \otimes_{\mathbb{C}} A) \otimes_A \mathbb{C} \simeq \mathfrak{g}_0$ as $\mathbb{C}$-modules. The morphisms can be given by

$$f : (\mathfrak{g}_0 \otimes_{\mathbb{C}} A) \otimes_A \mathbb{C} \to \mathfrak{g}_0, \qquad f(x \otimes a \otimes \alpha) = \alpha\epsilon(a)x = \alpha(\mathrm{id} \otimes \epsilon)(x \otimes a)$$

and

$$g : \mathfrak{g}_0 \to (\mathfrak{g}_0 \otimes_{\mathbb{C}} A) \otimes_A \mathbb{C}, \qquad g(x) = x \otimes 1 \otimes 1.$$

Using that the A-module structure on $\mathbb{C}$ is given by the augmentation ϵ, it is easy to see that f and g are mutually inverse. Let us show that f is a morphism of Lie algebras:

$$\begin{aligned} f([x \otimes a \otimes \alpha, y \otimes b \otimes \beta]) &= f([x \otimes a, y \otimes b]_\lambda \otimes \alpha\beta) \\ &= f([x \otimes 1, y \otimes 1]_\lambda(ab) \otimes \alpha\beta) \\ &= f([x \otimes 1, y \otimes 1]_\lambda \otimes \epsilon(ab)\alpha\beta) \\ &= \epsilon(ab)(\alpha\beta)(\mathrm{id} \otimes \epsilon)([x \otimes 1, y \otimes 1]_\lambda) \\ &= \epsilon(ab)(\alpha\beta)[x, y], \end{aligned}$$

where we used conditions (1) and (2), and on the other hand

$$[f(x \otimes a \otimes \alpha), f(y \otimes b \otimes \beta)] = \epsilon(ab)\alpha\beta[x, y].$$

This ends the proof of the lemma. □

Thus, the notion of general global deformations includes the deformations of product type, as it should. The reason for considering these more general deformations is that in general the Lie algebra deformations of a Lie algebra of vector fields coming from deformations of the underlying variety are not of product type.

2. A map from families of schemes to deformations of Lie algebras

A (n affine) *deformation* of a scheme X_0 over $\mathbb{C}$ is a scheme X over a $\mathbb{C}$-algebra A and a closed point $0 \in \mathrm{Spec}(A)$ (with residue field $\mathbb{C}$) such that the fiber $(X)_0$ of X in 0 satisfies $(X)_0 \simeq X_0$ as $\mathbb{C}$-schemes. Let us denote by $\pi : X \to \mathrm{Spec}(A)$ the structure morphism. Usually we suppose furthermore that $\pi : X \to \mathrm{Spec}(A)$ is flat, quasi-compact, surjective and finitely presented. Recall that when π is of finite type and A noetherian, π is automatically of finite presentation. As one encounters non-noetherian bases in descent theory, finite presentation is the good hypothesis to impose (for example, in the realm of fppf topology). We then regard X as a *family of schemes* deforming X_0. By definition of the fiber, $(X)_0 \simeq X_0$ means $X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathbb{C}) \simeq X_0$.

In order to get started, let us suppose that the family is *affine*, i.e. $X = \mathrm{Spec}(B)$, and that X_0 is also affine, i.e. $X_0 = \mathrm{Spec}(B_0)$. We are interested in the Lie algebra of vector fields on X which are tangent to the fibers of π, i.e. in the Lie algebra of A-linear derivations of B, denoted $\mathrm{Der}_A(B)$.

The following lemma expresses the behavior of A-linear derivations of B under base change.

Lemma 1. *Let A be a $\mathbb{C}$-algebra, and A' and B be A-algebras. Let $B' := B \otimes_A A'$ the coproduct in this situation. Suppose that B' is a flat B-module and that the B-module of Kähler differentials $\Omega_A(B)$ has a finite presentation. Then*

$$\mathrm{Der}_{A'}(B') \simeq \mathrm{Der}_A(B) \otimes_A A'$$

as A'-modules.

Remark 1. Translating into scheme language, i.e. $X = \mathrm{Spec}(A)$, $X' = \mathrm{Spec}(A')$, $Y = \mathrm{Spec}(B)$, $Y' = \mathrm{Spec}(B')$ is the fibered product $Y' = X' \times_X Y$, and the hypotheses are satisfied in case the induced morphisms $\pi : Y \to X$ and $X' \to X$ are of finite presentation and flat. This situation is called base change (in the fppf topology).

Proof. This follows in a straightforward manner from the corresponding base change for the modules of Kähler differentials. Here is an explicit proof:

The A'-module of Kähler differentials $\Omega_{B'/A'}$ of the A'-algebra B' represents the functor $\mathrm{Der}_{A'}(B', -)$ which implies in particular an isomorphism of B'-modules

$$\mathrm{Der}_{A'}(B') \simeq \mathrm{Hom}_{B'}(\Omega_{B'/A'}, B').$$

The hypothesis $B \otimes_A A' = B'$ implies $\Omega_{B'/A'} \simeq \Omega_{B/A} \otimes_B B'$, which in turn gives

$$\mathrm{Hom}_{B'}(\Omega_{B'/A'}, B') \simeq \mathrm{Hom}_{B'}(\Omega_{B/A} \otimes_B B', B').$$

Then we apply the "change of rings" isomorphism (meaning the adjointness of $- \otimes_B B'$ and $\mathrm{Hom}_{B'}(B', -)$)

$$\mathrm{Hom}_{B'}(\Omega_{B/A} \otimes_B B', B') \simeq \mathrm{Hom}_B(\Omega_{B/A}, B').$$

In order to retranslate the result into $\mathrm{Der}_A(B) \otimes_B B'$, we need for a B-module M a natural isomorphism

$$\mathrm{Hom}_B(M, B') \simeq \mathrm{Hom}_B(M, B) \otimes_B B'.$$

There is obviously such an isomorphism for $M = B$. Furthermore, the claim holds true for a B-module M of finite presentation as is easily checked (B' is a flat B-module by hypothesis). But $\Omega_{B/A}$ has a finite B-presentation (here we also use the noetherian hypothesis). Thus we can conclude

$$\mathrm{Hom}_B(\Omega_{B/A}, B') \simeq \mathrm{Hom}_B(\Omega_{B/A}, B) \otimes_B B',$$

and

$$\mathrm{Hom}_B(\Omega_{B/A}, B) \otimes_B B' \simeq \mathrm{Der}_A(B) \otimes_B B'.$$

Finally

$$\mathrm{Der}_A(B) \otimes_B B' = \mathrm{Der}_A(B) \otimes_B (B \otimes_A A') \simeq \mathrm{Der}_A(B) \otimes_A A'. \quad \square$$

Remark 2. It is easy to check that the above isomorphism

$$\mathrm{Der}_{A'}(B') \simeq \mathrm{Der}_A(B) \otimes_A A'$$

is an isomorphism of A'-Lie algebras when we give the right-hand side the bracket $[x \otimes a, y \otimes b] := [x, y] \otimes ab$. This follows from the fact that the bracket of derivations D, D' on both sides is given by $[D, D'] = D \circ D' - D' \circ D$.

Lemma 2. *Let A be a $\mathbb{C}$-algebra, $B \to A$ be an A-algebra, and $A \to \mathbb{C}$ be a $\mathbb{C}$-point of the affine scheme* $\mathrm{Spec}(A)$. *Suppose that the B-module $\Omega_{B/A}$ is projective. Then there is an isomorphism of $\mathbb{C}$-modules*

$$\mathrm{Der}_A(B) \otimes_A \mathbb{C} \simeq \mathrm{Der}_{\mathbb{C}}(B_0).$$

Proof. This is a slightly different situation from that in Lemma 3 where the module of derivations of the fiber (product) is just the tensor product of the initial module of derivations.

Its justification stems from the fact that in the above proof of Lemma 3, the natural isomorphism

$$\mathrm{Hom}_B(M, B') \simeq \mathrm{Hom}_B(M, B) \otimes_B B'$$

holds not only for $M = B^n$, but also for a direct factor of B^n, i.e. for each finitely generated projective B-module, by additivity of the concerned functors. Observe that we do not need any flatness assumption here. $\square$

Remark 3. Translated into scheme language, this lemma shows that the Lie algebra of derivations of a fiber of $\pi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is the tensor product of $\mathrm{Der}_A(B)$ by $\mathbb{C}$, seen as an A-module via the augmentation $A \to \mathbb{C}$, in case A is smooth. More generally, this is true at each smooth point of A.

Translated into the language of deformations exposed in Section 1, it shows that deformations of the underlying affine scheme $X_0 = \mathrm{Spec}(B_0)$ give rise to general global deformations of the Lie algebra of derivations $\mathrm{Der}(B_0)$.

In conclusion, there is a map, which we call I, assigning to an affine family of schemes X (over A) deforming a given affine scheme X_0 a general global A-deformation of the Lie algebra $\mathrm{Vect}(X_0)$.

Remark 4. Note that the map I does not define a functor in general, as $\mathrm{Der}_A(B)$ is not functorial in B; the point is that pushforward or pullback of vector fields along morphisms works for isomorphisms.

For example, one can define a genuine functor when restricting the source category to a groupoid of some kind of varieties with allowed morphisms being the only isomorphisms.

Corollary 1. *In the case of a product $Y' = Y \times X'$ and $X = \mathrm{Spec}(\mathbb{C})$, i.e. a trivial deformation, one gets an isomorphism of Lie algebras*

$$\mathrm{Der}_{A'}(B') \simeq \mathrm{Der}_{\mathbb{C}}(B) \otimes_{\mathbb{C}} A'.$$

Remark 5. Here the right-hand side carries the current algebra bracket $[x \otimes a, y \otimes b] := [x, y] \otimes ab$ which is considered as the trivial deformation, thus we see that trivial deformations of the underlying variety give rise to trivial global deformations of product type.

Remark 6. The difference between the notions of general global deformations and of global deformations of product type is that the latter considers only global deformations which are, as A-modules, tensor products with A over $\mathbb{C}$. But these are then free A-modules, whereas in general the Lie algebra of A-linear derivations on an algebra B has no reason to be a free A-module. It would be interesting to have a clear criterion when this is the case.

Nevertheless, in case the special point $0 \in \mathrm{Spec}(A)$ is a generic point in the moduli space $\mathcal{M}_{g,n}$, $\mathrm{Der}_A(B)$ is a free A-module, because in this situation, the sheaf of Krichever–Novikov algebras is locally free, and the local basis trivializes the module, see [19] p. 743. We owe this remark to M. Schlichenmaier.

3. Extracting affine families from proper families

Let S be some scheme over $\mathbb{C}$ and n, g integers such that $2g - 2 + n > 0$. A proper flat (surjective) family $\pi : C \to S$ whose geometric fibers C_s are reduced connected curves of genus $g = \dim H^1(C_s, \mathcal{O}_{C_s})$ with at most ordinary double points is called a *stable curve* of genus g over S. An *n-pointed stable curve* is a stable curve of genus g over S together with n sections $\sigma_1, \ldots, \sigma_n : S \to C$ of π such that the images $P_i = \sigma_i(s)$ are disjoint and smooth for all $i = 1, \ldots, n$ and all $s \in S$. Furthermore, the number of points where a non-singular rational component E of C_s meets the rest of C_s plus the number of points P_i on E must be at least 3, cf [16] Definition 1.1 p. 162. In the same vein, we call an *affine stable curve* a flat (surjective) family $\pi : C \to S$ whose geometric fibers are affine stable reduced and connected curves of genus g with at most ordinary double points. All our affine stable curves will be obtained from n-pointed stable curves by extracting the marked points and one may take this for the definition of an affine stable curve. We will then be in a position to apply the results of Section 2:

Lemma 3. *Let $\pi : C \to S$ be a punctured stable curve, and $\sigma_1, \ldots, \sigma_n$ the corresponding sections of π. Denote by $\triangle$ the union $\triangle = \bigcup_{i=1}^n \sigma_i(S)$. Then $\pi : C \setminus \triangle \to S$ is an affine stable curve and an affine morphism.*

Proof. It is shown in [16] p. 173 Corollary 1.9 that some tensor power of the sheaf $\omega_{C_s/\mathbb{C}}(P_1 + \cdots + P_n)$ is very ample, where $\omega_{C_s/\mathbb{C}}$ is the dualizing sheaf of C_s over $\mathbb{C}$. Here we denote by $C_s = \pi^{-1}(s)$ for $s \in S$, and $P_i = \sigma_i(s)$ for $i = 1, \ldots, n$. Let us denote by $\mathcal{L}_\triangle$ the line bundle $\omega_{C/S}(\triangle)$ on C. By Grauert's theorem (cf [14] pp. 288–289), we have $R^i\pi_*(\mathcal{L}_\triangle^{\otimes n}) = 0$ for all $i > 0$ and $n \gg 0$. The local to global spectral sequence gives therefore $H^i(C, \mathcal{L}_\triangle^{\otimes n}) = 0$ for all $i > 0$ and $n \gg 0$. $\mathcal{L}_\triangle$ is then an ample line bundle on C, and we use some power $\mathcal{L}_\triangle^{\otimes k}$ of $\mathcal{L}_\triangle$ to embed C into projective space $\mathbb{P}_A^N$; let $i : C \to \mathbb{P}_A^N$ be the corresponding embedding. Then $C \setminus \triangle$ maps under i into the complement of a hyperplane, and $i|_{C\setminus\triangle}$ is affine. The claim follows. □

Now we pass on to moduli space; there are (at least) two points of view on the moduli space of curves, and it is the necessity of the situation which determines which one to take. One point of view is the following (the GIT or coarse moduli space point of view): the *moduli space* (of stable connected projective algebraic curves over $\mathbb{C}$ of fixed genus g with n marked points) is a scheme whose geometric points represent the isomorphism classes of stable punctured curves. We observe in Lemma 4 below that isomorphic punctured curves give rise to isomorphic deformations, and thus obtains the fact that I factors to a map from the set of S-points of the moduli space to global deformations of vector field Lie algebras over S.

More precisely, let $\pi : C \to S$ and $\pi' : C' \to S$ be two isomorphic punctured curves by an S-morphism ϕ which respects the punctures. Let $U = \mathrm{Spec}(B) \subset C$ be an affine open set such that $\pi|_U$ and $\pi|_{\phi(U)}$ are affine families of schemes. Denote by B and B' the corresponding algebras, i.e. $U = \mathrm{Spec}(B)$ and $\phi(U) = \mathrm{Spec}(B')$. One may assume that $\pi(U) = \pi'(\phi(U)) = \mathrm{Spec}(A)$.

Lemma 4. *In the above notation, the isomorphic curves $\pi : C \to S$ and $\pi' : C' \to S$ give rise to isomorphic Lie algebras* $\mathrm{Der}_A(B)$ *and* $\mathrm{Der}_A(B')$.

On the other hand, let $\pi : C \to S$ and $\pi' : C' \to S$ be two punctured curves, let $T = \mathrm{Spec}(A) \subset S$ be an affine open subset of the parameter space and denote by $U = \mathrm{Spec}(B)$ and $U' = \mathrm{Spec}(B')$ the affine open sets $U = \pi^{-1}(T)$ and $U' = (\pi')^{-1}(T)$ corresponding to the $\triangle$ (resp. $\triangle'$) complement in C (resp. C').

Theorem 1. *In the above notation, suppose*

- *either that B and B' are integral, normal, of finite type over A noetherian,*
- *or that all fibers of C and C' are smooth curves,*
- *or that* $\mathrm{Der}_A(B)$ *(resp.* $\mathrm{Der}_A(B')$*) is a free B (resp. B') module of rank 1.*

Then $\mathrm{Der}_A(B) \simeq \mathrm{Der}_A(B')$ *as Lie algebras over A imply that there exists a T-isomorphism $\phi : U \to U'$ which is compatible with the punctures.*

Remark 7. The special case where T is a closed point of residue field $\mathbb{C}$ is easily solved: the claim is then the fact that an isomorphism between punctured projective curves $f : (X \setminus \{p_1, \ldots, p_n\}) \simeq (Y \setminus \{q_1, \ldots, q_n\})$ can be extended to an isomorphism $X \simeq Y$ respecting the punctures (use Proposition 6.8 in [14] p. 43 and induction applied to $i \circ f : (X \setminus \{p_1, \ldots, p_n\}) \to (Y \setminus \{q_1, \ldots, q_n\}) \hookrightarrow Y$).

The proof of the above theorem follows from Pursell–Shanks theory, and is given in the next section.

One deduces from the theorem that we get a well-defined *injective* map from the set of S-points of the moduli space of punctured smooth curves (deforming a fixed punctured curve $(X_0, p_1, \ldots, p_n)$) to the set of isomorphism classes of deformations of the Lie algebra $\mathrm{Vect}(X_0 \setminus \{p_1, \ldots, p_n\})$ with base S.

4. Pursell–Shanks theory

In this section, we develop another key ingredient of our study of the moduli space of deformations of a given Lie algebra of regular vector fields on an affine curve. Here we recall how the Lie algebra of vector fields on a manifold encodes geometric objects of the manifold. This theory is due to Pursell–Shanks, Omori, Amemiya [2], Grabowski [10], and in our framework to Siebert [20].

To give an example of the kind of proposition studied in this theory, Siebert proves that two complex normal reduced irreducible affine algebraic varieties $Y = \mathrm{Spec}(B)$ and $Y' = \mathrm{Spec}(B')$ over $\mathbb{C}$ are isomorphic if and only if their Lie algebras of vector fields are isomorphic as Lie algebras (over $\mathbb{C}$). The method of the proof describes the points of the underlying variety in terms of data attached to the Lie algebra of vector fields $\mathcal{L}$. Indeed, he proves for irreducible Y a bijection between the regular points of Y and the finite-codimensional maximal subalgebras L of $\mathcal{L}$ such that L does not contain a proper Lie-ideal. Call the set of these subalgebras $\mathcal{M}$. The idea is that such an L is the Lie subalgebra of vector fields vanishing at the point it represents. This bijection can be used to express the Zariski topology on Y in Lie algebraic terms. Then Siebert defines an algebra of functions $B(\mathcal{L})$ (cf Definition 10 [20]) on Y by taking those functions $f : \mathcal{M} \to \mathbb{C}$ such that for all $\delta \in \mathcal{L}$, there is a $\theta \in \mathcal{L}$ such that $f(L)\delta - \theta \in L$ for all $L \in \mathcal{M}$. $B(\mathcal{L})$ thus encodes the coefficient functions of the vector fields. Siebert shows that for $\mathcal{L} = \mathrm{Der}_{\mathbb{C}}(B)$ with $Y = \mathrm{Spec}(B)$ irreducible of finite type over $\mathbb{C}$, $B(\mathcal{L})$ is an integral extension of B. The normality of B then forces $B(\mathcal{L}) = B$. In this sense, Siebert does not construct an explicit isomorphism $Y \simeq Y'$, but he deduces the *existence* of an isomorphism from the translation of the geometric structures into Lie algebraic objects.

Our idea is that the above theory reflects that the map I is injective in a certain sense. For this we need a *relative version* of the above statement, namely $\mathrm{Der}_A(B) \simeq \mathrm{Der}_A(B')$ if and only if $\mathrm{Spec}(B) \simeq \mathrm{Spec}(B')$ by an isomorphism over $\mathrm{Spec}(A)$. The following statement will be the main result of this section.

Proposition 2. *Suppose*

- *either that B and B' are integral, normal, of finite type over A noetherian,*
- *or that all fibers of C and C' are smooth (cf notations Section 3),*
- *or that* $\mathrm{Der}_A(B)$ *(resp.* $\mathrm{Der}_A(B')$*) is a free B (resp. B') module of rank 1.*

Then $\mathrm{Der}_A(B) \simeq \mathrm{Der}_A(B')$ *if and only if* $\mathrm{Spec}(B) \simeq \mathrm{Spec}(B')$ *by an isomorphism over* $\mathrm{Spec}(A)$.

Proof. Denote as before $\mathrm{Der}_A(B) = \mathcal{L}$. It is a modular Lie algebra in the sense of Definition 1 of [20] (with respect to B). To every algebra homomorphism $\phi : A \to \mathbb{C}$, i.e. to every $\mathbb{C}$-point x of $\mathrm{Spec}(A) = X$, we associate the fiber $\pi^{-1}(x)$ of $\pi : Y = \mathrm{Spec}(B) \to X$. Now we apply Siebert's theory to describe the smooth fibers $\pi^{-1}(x)$: ϕ gives rise to the $\mathbb{C}$-algebra $B \otimes_A \mathbb{C}$ where A acts on $\mathbb{C}$ via ϕ and on B via $A \to B$. For each point $\phi = x \in X$, there is a projection

$$\pi_\phi : \mathrm{Der}_A(B) \to \mathrm{Der}_A(B) \otimes_A \mathbb{C} \to \mathrm{Der}_A(B \otimes_A \mathbb{C})$$

given by the composition of $X \mapsto X \otimes 1$ and

$$D \otimes 1 \mapsto (b \otimes 1 \mapsto D(b) \otimes 1).$$

In this way we reduce at a point x the problem to the fiber $\pi^{-1}(x)$, and Siebert's theory implies for $\pi^{-1}(x)$ smooth that

$$\pi^{-1}(x) = \mathcal{M}_\phi,$$

the space of maximal, finite-codimensional subalgebras L_ϕ of $\mathcal{L}_\phi := \mathrm{Der}_{\mathbb{C}}(B \otimes_A \mathbb{C})$, and furthermore that the algebra of regular functions $\mathrm{Reg}(\pi^{-1}(x))$ is

$$\mathrm{Reg}(\pi^{-1}(x)) = \mathcal{B}_\phi$$

with

$$\mathcal{B}_\phi := \{f \in \mathbb{C}^{\mathcal{M}_\phi} \mid \forall \delta \in \mathcal{L}_\phi \exists \theta \in \mathcal{L}_\phi : f(L_\phi)\delta - \theta \in L_\phi \ \forall L_\phi \in \mathcal{M}_\phi\}.$$

In order to describe not only the fibers, but all points of Y, consider now the set of pairs

$$\mathcal{M} = \{(\phi, L_\phi) \mid \phi = x \in X, \text{ and } L_\phi \in \mathcal{M}_\phi\}$$

and

$$\mathcal{B} = \{f \in \mathbb{C}^{\coprod_\phi \mathcal{M}_\phi} \mid \forall \delta \in \mathcal{L} \exists \theta \in \mathcal{L} : \pi_\phi(f(L_\phi)\delta - \theta) \in L_\phi \ \forall L_\phi \in \mathcal{M}_\phi \text{ and } \forall \phi = x \in X\}.$$

Observe that we first have to project the expression $f(L_\phi)\delta - \theta$ onto the fiber via π_ϕ in order to state that the result should vanish at L_ϕ, seen as a point of $\mathcal{M}_\phi$.

Let us show that $\mathcal{B}$ is an integral extension of B. Indeed, clearly $B \subset \mathcal{B}$, because $\mathcal{L}$ is a B-module.

If B is a finitely generated A-algebra, there exists an epimorphism

$$B_n := A[X_1, \ldots, X_n] \to B,$$

and consequently a monomorphism

$$\mathrm{Der}_A(B) \to \mathrm{Der}_A(B_n) \cong B^n.$$

Denote by ∂_j (the isomorphic images in $\mathrm{Der}_A(B_n)$ of) the generators of the free module B^n. For a given $f \in \mathcal{B}$, choose δ and θ such that the condition to be in $\mathcal{B}$ holds, and express $\delta = \sum_j g_j \partial_j$ and $\theta = \sum_j h_j \partial_j$. Abusing slightly the notation, we have

$$f(p) g_j(p) = h_j(p)$$

for all $p \in \pi^{-1}(x)$, by definition of δ and θ. This means that f is a rational function on Y, regular on fibers $\pi^{-1}(x)$ for x smooth. Therefore $\mathcal{B} \subset K(Y)$. $\mathcal{L}$ is a $\mathcal{B}$-module by setting $f \cdot \delta = \theta$, for the θ specified through the definition of $\mathcal{B}$ by f and δ. Its annihilator is zero. On the other hand, $\mathcal{L} = \mathrm{Der}_A(B)$ is a finitely generated B-module if A is noetherian. By a well-known criterion (cf [6] p. 123, cor. 4.6), $\mathcal{B}$ is thus an integral extension of B.

In conclusion, if Y is normal, $B = \mathcal{B}$ is characterized by Lie theoretic methods, and the first part of the proposition follows.

If all the fibers are smooth, $\mathcal{B}$ is even more readily identified with B. If $\mathcal{L}$ is a free rank 1 B-module, say with generator δ, we have for $f \in \mathcal{B}$ by definition $f\delta = \theta$, but on the other hand $f'\delta = \theta$ for some $f' \in B$. It follows $f = f'$ on all $\mathcal{M}_\phi$, which permits us to conclude this case. □

Remark 8. In the smooth case, one can also argue in several different ways: in case $Y = \mathrm{Spec}(B)$, $X = \mathrm{Spec}(A)$ and $\pi : Y \to X$ are smooth and $T\pi$ is surjective, the tangent bundles $\mathcal{T}_Y$ of Y, and $\mathcal{T}_X$ of X fit into an exact sequence

$$0 \to \mathcal{T}_{Y/X} \to \mathcal{T}_Y \to \pi^*\mathcal{T}_X \to 0,$$

defining the relative tangent bundle $\mathcal{T}_{Y/X}$ whose regular sections are $\mathrm{Der}_A(B)$. One can show that $\mathrm{Der}_A(B)$ is an admissible B-Lie module in the same way Amemiya showed it for Lie algebras defining pseudo foliations.

One can also show directly that $\mathrm{Der}_A(B)$ is "strongly nowhere vanishing" in the sense of Grabowski, in the smooth setting.

Let us finish the *proof of* Theorem 1: it remains to show that two families of punctured curves $\pi : C \to S$ and $\pi' : C' \to S$ with associated divisors $\triangle$ and $\triangle'$ such that $\phi : C \setminus \triangle \cong C' \setminus \triangle'$ as schemes over $\mathrm{Spec}(A)$ are isomorphic as schemes over $\mathrm{Spec}(A)$. Indeed, a point where ϕ is not defined is on some fiber of π (the families are surjective), but there the restriction of ϕ defines a rational map of curves, thus there is no such point.

5. The stack of deformations

It is well-known that the moduli space of curves carries a structure of an algebraic *scheme* (the quasi-projective irreducible coarse moduli scheme), or the structure of an *algebraic stack*. The first approach follows from geometric invariant theory (GIT) and has the advantage that one stays within the framework of standard algebraic geometry. The second approach stems from Grothendieck's program of characterizing the functor of points $h_X : \mathtt{Sch} \to \mathtt{Sets}$, $S \mapsto \mathrm{Hom}_{\mathtt{Sch}}(S, X)$ associated to a fixed scheme X within all functors $F : \mathtt{Sch} \to \mathtt{Sets}$. Some necessary conditions generalized from this example lead to the definition of a $\mathbb{C}$-stack, a device which generalizes schemes and which is important in the study of classification problems in algebraic geometry where the "classifying space" is not a scheme any more. Indeed, in the category of (algebraic) stacks, the moduli functor is representable, thus it does not only possess a coarse, but a fine moduli stack. The disadvantage of the stack approach is that one has to (re)learn how to do (algebraic) geometry with stacks instead of schemes, varieties or manifolds. One advantage of the stack approach is that it includes (at least formally) the approach by coarse moduli spaces, cf Remark (3.19) [17]. The GIT is used in the context of stacks in order to show that some stacks are algebraic.

Our understanding of the notion of algebraic or differentiable stacks is based on [5,12,18,21].

When we address the question about some kind of structure of a variety on the space of deformations, it will lead naturally to the question whether the map I respects these structures.

Let us define a stack of deformations of Lie algebras. It is some kind of functor, more precisely a lax functor, which does not associate to an affine scheme $\mathrm{Spec}(A)$ the set of isomorphism classes of A-Lie algebras, but rather the groupoid of A-Lie algebras, keeping the data of the iso and automorphisms between/of A-Lie algebras. There are (at least) two points of view on stacks; let us use here the one based on lax functors (or 2-functors).

As in any geometric theory, one first needs to fix a class of standard spaces to which the spaces one wants to define should be locally isomorphic. Led by the example of the category of open sets of a manifold (the ones isomorphic to $\mathbb{R}^n$ serving exactly the expressed need), one defines a Grothendieck (pre-)topology on a category $\mathtt{C}$ to be a collection of covering families $T(U)$ for each object U of $\mathtt{C}$ such that covering families contain isomorphisms, are stable under base change, and are stable under refinement. The category of interest for us will be $(\mathtt{Aff}/\mathbb{C})$, the category of affine schemes over $\mathbb{C}$, and as Grothendieck topologies on $(\mathtt{Aff}/\mathbb{C})$, we will regard the classical four topologies fpqc, fppf, étale and Zariski. The preceding abbreviations mean *fidèlement plat, quasi-compact*, and *fidèlement plat, de présentation finie* respectively. This indicates the conditions on a morphisms between affine schemes to be a member of the covering family. We refer to [1] for basics about faithfully flat or quasi-compact morphisms, or morphisms of finite presentation. The four topologies are ordered from finest (fpqc) to coarsest (Zariski). See also the precisions made by Kleiman and Vistoli on the fpqc topology in [21].

A *lax functor* associates to an affine scheme a groupoid. This association is not an ordinary functor, but a 2-functor. Here the category $\mathtt{Aff}/\mathbb{C}$ is seen in a trivial way as a 2-category (the 2-morphisms are identities between compositions of 1-morphisms), and the 2-functor has values in the 2-category of groupoids.

More precisely, we define the *lax deformation functor* $\mathcal{D}\mathrm{ef}$ in the following way: let $U = \mathrm{Spec}(B) \in \mathrm{ob}(\mathtt{Aff}/\mathbb{C})$ be some affine scheme over $\mathbb{C}$. The lax functor $\mathcal{D}\mathrm{ef} : (\mathtt{Aff}/\mathbb{C}) \to \mathtt{Gpd}$ associates to U the groupoid $\mathcal{D}\mathrm{ef}(U)$ having as class of objects the B-Lie algebras and as morphisms only isomorphisms of B-Lie algebras. We forget for the moment that the Lie algebra should be isomorphic to a given one at a specified point. For a morphism $f : U' \to U$ in $(\mathtt{Aff}/\mathbb{C})$, we denote by $\tilde{f} : B \to B'$ the corresponding morphism of $\mathbb{C}$-algebras. To f, the lax functor $\mathcal{D}\mathrm{ef}$ associates a functor $f^* : \mathcal{D}\mathrm{ef}(U) \to \mathcal{D}\mathrm{ef}(U')$ given by $f^*(\mathfrak{g}) = \mathfrak{g} \otimes_B B'$ (where the bracket on the tensor product is the current algebra bracket) for any B-Lie algebra $\mathfrak{g}$. Here B' is seen as a B-algebra via $\tilde{f}$.

Let us verify the axioms of a lax functor (2-functor): let $g : U'' \to U'$ be another morphism in $(\mathtt{Aff}/\mathbb{C})$ with associated morphism $\tilde{g} : B' \to B''$ of $\mathbb{C}$-algebras. Then we have $g^* \circ f^* \simeq (f \circ g)^*$. Indeed, $(f \circ g)^*(\mathfrak{g}) = \mathfrak{g} \otimes_B B'' \simeq (\mathfrak{g} \otimes_B B') \otimes_{B'} B'' = g^*(f^*(\mathfrak{g}))$, where B'' (resp. B') is seen as a B', B (resp. B-) module via $\tilde{g}$, $\widetilde{(f \circ g)} = \tilde{g} \circ \tilde{f}$ (resp. $\tilde{f}$).

Now consider a third morphism $h : U''' \to U''$ ($\tilde{h} : B'' \to B'''$). We have then a commutative diagram of B'''-Lie algebras

$$\begin{array}{ccc} h^* \circ g^* \circ f^*(\mathfrak{g}) = \big((\mathfrak{g} \otimes_B B') \otimes_{B'} B''\big) \otimes_{B''} B''' & \longrightarrow & (\mathfrak{g} \otimes_B B'') \otimes_{B''} B''' \\ \downarrow & & \downarrow \\ (g \circ h)^* \circ f^*(\mathfrak{g}) = (\mathfrak{g} \otimes_B B') \otimes_{B'} B''' & \longrightarrow & \mathfrak{g} \otimes_B B''' \end{array}$$

Let us show now that our lax functor is a $\mathbb{C}$-stack. It is here that the chosen Grothendieck topology on $\mathtt{Aff}/\mathbb{C}$ comes into play. In order to be a $\mathbb{C}$-stack, the lax functor should be on the one hand a sheaf of spaces, and on the other hand, one should be able to define in a unique way morphisms and objects from data given on a covering family (i.e. equivalence of descent categories). More precisely, the first condition is that for all $U \in \mathrm{ob}(\mathtt{Aff}/\mathbb{C})$, and all $\mathfrak{g}, \mathfrak{h} \in \mathrm{ob}(\mathcal{D}\mathrm{ef}(U))$, the presheaf

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{h}) : \mathtt{Aff}/\mathbb{C} \to \mathtt{Sets},$$

given by

$$(f : U' \to U) \mapsto \mathrm{Hom}_{\mathcal{D}\mathrm{ef}(U')}(f^*\mathfrak{g}, f^*\mathfrak{h}),$$

should be a sheaf, i.e. one should have an exact sequence of sets

$$\mathrm{Hom}_{\mathcal{D}\mathrm{ef}(U)}(\mathfrak{g}, \mathfrak{h}) \longrightarrow \mathrm{Hom}_{\mathcal{D}\mathrm{ef}(U')}(f^*\mathfrak{g}, f^*\mathfrak{h}) \rightrightarrows \mathrm{Hom}_{\mathcal{D}\mathrm{ef}(U'')}(q^*\mathfrak{g}, q^*\mathfrak{h}),$$

which lies over the "exact" sequence of objects in $\mathtt{Aff}/\mathbb{C}$

$$U'' = U' \times_U U' \rightrightarrows U' \longrightarrow U,$$

or, equivalently, over the "exact" sequence of $\mathbb{C}$-algebras

$$B \longrightarrow B' \rightrightarrows B'' = B' \otimes_B B',$$

q being associated to the arrow $\tilde{q} : B \to B''$. Let us emphasize that $f : U' \to U$ is here a covering family in the given Grothendieck topology, which means that the morphism f has some geometric properties, but should be moreover surjective (because one thinks U' to be $\coprod_i U'_i$, and the U'_i should cover U).

The second condition is that any descent data is *effective*, i.e. that for any covering family $f : U' \to U$, the category $\mathcal{D}\mathrm{ef}(U)$ is equivalent to the category of descent data in $\mathcal{D}\mathrm{ef}(U')$, i.e. to the category whose objects are the pairs (x', ϕ) of an object $x' \in \mathrm{ob}(\mathcal{D}\mathrm{ef}(U'))$ and an isomorphism $\phi : p_1^* x' \to p_2^* x'$, where $p_1, p_2 : U'' \to U'$ are the two projections, such that ϕ satisfies the cocycle identity $p_{13}^*\phi = p_{23}^*\phi \circ p_{12}^*\phi$, where p_{ij} are projections $p_{ij} : U''' \to U''$ onto a choice of two factors in the triple fibered product.

It is a non-trivial fact that the lax functor associating to an affine scheme $U = \mathrm{Spec}(B)$ the category of B-modules is a $\mathbb{C}$-stack where $\mathtt{Aff}/\mathbb{C}$ carries the fpqc topology. This is the content of Grothendieck's theorem of *descente fidèlement plate* [12]. One easily deduces (cf Exposé VIII, Section 2 in [12]) the following

Theorem 2. *The deformation lax functor* $\mathcal{D}\mathrm{ef}$ *is a* $\mathbb{C}$*-stack.*

In this form, $\mathcal{D}\mathrm{ef}$ is certainly not an algebraic stack, nor a differentiable stack in the sense of [18], as one needs a covering (algebraic) space. Nevertheless, fixing an infinite-dimensional Lie algebra $\mathrm{Der}_A(B)$ with $X_0 = \mathrm{Spec}(B)$, which is to be deformed, one can restrict $\mathcal{D}\mathrm{ef}$ to the $\mathbb{C}$-stack of deformations of $\mathrm{Der}_A(B)$ by considering first the full subcategory $\mathtt{Aff}_{X_0}/\mathbb{C}$ of affine schemes which have X_0 as fiber over a closed point. Let $f : \mathtt{Aff}_{X_0} \to \mathtt{Aff}$ be the inclusion functor. Then one takes the direct image [11] p. 83, [18] p. 16 $f_*\mathcal{D}\mathrm{ef}$ as the deformation lax functor, and it is a standard matter to prove that $f_*\mathcal{D}\mathrm{ef}$ is still a $\mathbb{C}$-stack (in the induced Grothendieck topology on $\mathtt{Aff}_{X_0}$). This stack $f_*\mathcal{D}\mathrm{ef}$ is a more natural candidate for being an algebraic stack. It would be interesting to examine whether the miniversal deformation space [7] of a Lie algebra $\mathfrak{g} := \mathrm{Der}_A(B)$ with finite-dimensional $H^2(\mathfrak{g}, \mathfrak{g})$ could serve as such a a covering space in order to show that in this situation, $f_*\mathcal{D}\mathrm{ef}$ is an algebraic stack.

6. The morphism *I*

In this section, we will show that the map I gives a morphism of $\mathbb{C}$-stacks from the moduli stack $\mathcal{M}_{g,n}$ to the deformation stack $\mathcal{D}\mathrm{ef}$, at least in any topology which is more coarse than the fppf topology (for example, in the étale topology, in which the moduli stack is usually considered).

Theorem 3. *The map* I *defines a morphism of* $\mathbb{C}$*-stacks from* $\mathcal{M}_{g,n}$ *to* $\mathcal{D}\mathrm{ef}$ *in any topology which is more coarse than the fppf topology.*

Proof. Recall that the *moduli stack* $\mathcal{M}_{g,n}$, regarded as a lax functor, associates to an affine scheme S over $\mathbb{C}$ the groupoid of proper flat families of curves $\pi : C \to S$ with sections $\sigma_1, \ldots, \sigma_n : S \to C$. The map I is now regarded as morphism of lax functors, i.e. for all affine schemes U,

$$I(U) : \mathcal{M}_{g,n}(U) \to \mathcal{D}\mathrm{ef}(U)$$

is the functor associating to the family of curves $\pi : C \to U$ with sections $\sigma_1, \ldots, \sigma_n : U \to C$ the Lie algebra $\mathrm{Der}_A(B)$ such that $\mathrm{Spec}(B) = C \setminus \triangle$ with $\triangle = \bigcup_{i=1}^n \sigma_i(U)$ and $U = \mathrm{Spec}(A)$. Observe that $I(U)$ is a functor here, because $\mathcal{M}_{g,n}(U)$ and $\mathcal{D}\mathrm{ef}(U)$ are groupoids. Now in order to have a morphism of stacks, one needs furthermore a compatibility on 2-morphisms, more precisely, for every covering family $f : U' \to U$ in $\mathtt{Aff}/\mathbb{C}$, there exists in the diagram

$$\begin{array}{ccc} \mathcal{M}_{g,n}(U) & \xrightarrow{I(U)} & \mathcal{D}\mathrm{ef}(U) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \mathcal{M}_{g,n}(U') & \xrightarrow{I(U')} & \mathcal{D}\mathrm{ef}(U') \end{array}$$

a natural isomorphism $\alpha(f) : f^* \circ I(U) \to I(U') \circ f^*$ between the two possible compositions of functors. This last statement is true in the fppf topology, because of Lemma 1, the compatibility of the fppf topology with base change, and the fact that

$$(X \times_U U') \setminus \triangle' \cong (X \setminus \triangle) \times_U U'$$

which follows from the compatibility of the embedding $X \hookrightarrow \mathbb{P}^N$ with base change. This concludes the proof of Theorem 3. $\square$

Remark 9. The morphism I is almost a monomorphism; indeed, for integral normal B we can apply Pursell–Shanks theory and use Theorem 1. But for a general affine base scheme $S = \mathrm{Spec}(A)$, one cannot hope to have monomorphy of I. For example, a non-reduced scheme and its reduction may well have isomorphic tangent Lie algebra, cf Remark on p. 313 in [15].

Theorem 1 shows however that I is a monomorphism for the stack of families of smooth curves.

7. Lie algebra cohomology sheaves on moduli stack

In this section, we will examine natural sheaves on the deformation stack $\mathcal{D}$ef arising from the cohomology of Lie algebras. The map I permits them to be pulled back to $\mathcal{M}_{g,n}$.

Let $\mathfrak{g}$ be an A-Lie algebra, where A is some $\mathbb{C}$-algebra. Let M be an A-module carrying an A-linear action of $\mathfrak{g}$. Then denote by $H^*_A(\mathfrak{g}; M)$ the cohomology of $\mathfrak{g}$ with values in M, where cochains are supposed to be A-linear. There is a base change formula for this kind of cohomology space:

Lemma 5. *Let $A \to B$ be a morphism of commutative unital $\mathbb{C}$-algebras such that B is an A-module of finite presentation. Then*

$$H^*_B(\mathfrak{g} \otimes_A B; M \otimes_A B) \cong H^*_A(\mathfrak{g}; M) \otimes_A B.$$

Proof. Recall that the cochain complex computing $H^*_A(\mathfrak{g}; M)$ consists of spaces $\mathrm{Hom}_A(\Lambda^p_A(\mathfrak{g}), M)$, where $\Lambda^p_A(\mathfrak{g})$ means the skewsymmetric tensor product over A of p factors of $\mathfrak{g}$. The proof is easily deduced from three steps:

$$(\mathfrak{g} \otimes_A B) \otimes_B (\mathfrak{g} \otimes_A B) \cong \mathfrak{g} \otimes_A \mathfrak{g} \otimes_A B,$$
$$\mathrm{Hom}_B(\mathfrak{g} \otimes_A \mathfrak{g} \otimes_A B, X) \cong \mathrm{Hom}_A(\mathfrak{g} \otimes_A \mathfrak{g}, X),$$

where X is a B-module, on the RHS viewed as an A-module, and

$$\mathrm{Hom}_A(\mathfrak{g} \otimes_A \mathfrak{g}, M \otimes_A B) \cong \mathrm{Hom}_A(\mathfrak{g} \otimes_A \mathfrak{g}, M) \otimes_A B,$$

where we used in the last step the finite presentation of B as an A-module. □

Corollary 2. *Let $B = \mathbb{C}$ and $A \to \mathbb{C}$ be a retraction of the unity $\mathbb{C} \to A$. Then*

$$H^*_{\mathbb{C}}(\mathfrak{g} \otimes_A \mathbb{C}; M \otimes_A \mathbb{C}) \cong H^*_A(\mathfrak{g}; M) \otimes_A \mathbb{C},$$

meaning that the fiber of the cohomology sheaf is the cohomology of the fiber of the family of Lie algebras, cf Lemma 2.

The above lemma means that the prescription

$$\mathfrak{g} \mapsto H^*_A(\mathfrak{g}; M)$$

for a fixed module M, defines a cartesian sheaf of vector spaces $H^*(M)$ on the stack $\mathcal{D}$ef in the fppf topology, and a cartesian sheaf $I^*(H^*(M))$ on the stack $\mathcal{M}_{g,n}$. The fixed module M is supposed to be an A-module and a $\mathfrak{g}$-module for any A-Lie algebra $\mathfrak{g}$. Examples are the ground field $M = \mathbb{C}$ ("trivial coefficients"), the Lie algebra itself $M = \mathfrak{g}$ ("adjoint coefficients"), or algebraic constructions with these (like the symmetric algebra $M = S^*\mathfrak{g}$ etc).

What is known about these cohomology spaces ?

It is known that $H^1(\mathbb{C}) = 0$ on the locus of smooth families, because the Lie algebra of vector fields on a smooth curve is simple, see [3] p. 21. I can also show that $H^2(Mer_k; \mathbb{C}) \cong H^1(\Sigma_k)$, the singular cohomology of the curve $\Sigma_k := \Sigma \setminus \{p_1, \ldots, p_k\}$ for any Lie algebra Mer_k of meromorphic vector fields on the compact connected Riemann surface Σ with possible pôles in the k fixed points $p_1, \ldots, p_k$. The adjoint cohomology seems to be trivial. All these are *algebraic* or *discrete* cohomology computations. For *continuous cohomology*, i.e. cohomology with continuous cochains, the Lie algebras of meromorphic vector fields Mer_k are endowed with the subspace topology from the Lie algebra $Hol(\Sigma_k)$ of all holomorphic vector fields on Σ_k. The continuous cohomology of $Hol(\Sigma_k)$ (all degrees and values in various modules) is known from the work of Kawazumi, but it seems artificial to endow these algebraic objects with topology.

One deduces from these results that the principle that cohomology should only increase in isolated points is not true in this infinite-dimensional setting. Indeed, Fialowski and Schlichenmaier [8] have a family of genus 0 curves with 3 points which degenerate to the *Witt algebra*

$$\mathrm{Vect}_{\mathrm{pol}}(S^1) = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} x^{i+1} \frac{\mathrm{d}}{\mathrm{d}x}.$$

It is known that $H^2(\mathrm{Vect}_{\mathrm{pol}}(S^1); \mathbb{C})$ is of dimension one, but on the other hand, $H^2(Mer_3; \mathbb{C})$ is of dimension two.

8. Deformations of Lie algebras arising from families of singular plane curves

There is a way to incorporate families having as some of their members non-stable curves such as cusps, for example, into the picture. Up to now, we took into account only families of singular curves which constitute the boundary of the moduli stack. The other approach is to normalize the family first in a certain way and to map the thus obtained family of smooth curves into $\mathcal{M}_{g,n}$. This corresponds to the normalization which occurs in the procedure defined by Fialowski and Schlichenmaier [8].

Consider families of irreducible plane curves with nodes and cusps as their only singularities. Arbitrary plane curves of degree d are parametrized by a Hilbert scheme $\mathrm{Hilb}(d)$ which can be identified with $\mathbb{P}^N$ for $N = d(d+3)/2$. Indeed, a curve is just represented by its homogeneous equation, up to scalars. $\mathrm{Hilb}(d)$ is a fine moduli space and possesses a universal family, cf [13].

In the following, we are inspired by the introduction of [9]. Let $\Sigma^d_{k,n} \subset \mathbb{P}^N$ be the closure in the Zariski topology of the set of reduced and irreducible plane curves of degree d with k cusps and n nodes. Let $\Sigma \subset \Sigma^d_{k,n}$ be an irreducible component of $\Sigma^d_{k,n}$. Denote by Σ_0 the open set of Σ of points $s \in \Sigma$ such that Σ is smooth at s and such that s corresponds to a reduced and irreducible plane curve of degree d with n nodes and k cusps and no further singularities. The restriction of the universal family $\mathcal{S}_0 \to \Sigma_0$ is an *equigeneric* family of curves, i.e. the genus g of the normalizations of the fibers of the family is constant. This property implies that one gets a family of smooth curves by normalizing the total space, cf Theorem 2.5 in [4]. In this way, there is a regular map of schemes from $\mathcal{S}_0$ to the scheme of moduli of curves, and a morphism of algebraic stacks $\mathcal{S}_0 \to \mathcal{M}_g$, where $\mathcal{S}_0$ is seen as an algebraic stack via its functor of points and $g = \binom{d-1}{2} - k - n$. One can introduce m marked points in order to get a morphism of algebraic stacks

$$N : \mathcal{S}^m_0 \to \mathcal{M}_{g,m}.$$

The composition

$$I \circ N : \mathcal{S}^m_0 \to \mathcal{M}_{g,m} \to \mathcal{D}ef$$

gives then the construction of Fialowski and Schlichenmaier in a conceptual way: the map N consists of normalizing, i.e. desingularizing the singular fiber, and the map I consists of extracting the m marked points from the family of smooth curves and then taking the Lie algebra of vector fields tangent to the fiber.

By its construction, it is clear that the same framework applies to flat equigeneric families of projective curves with all fibers reduced, where parameter and total space are reduced separated schemes of finite type over $\mathbb{C}$, see [4] pp. 436–437.

9. Examples

This section is based on discussions with Martin Schlichenmaier.

(1) Let us first consider a rather trivial example. Let $\mathfrak{g}$ be a non-abelian Lie algebra with a Lie bracket $[\,,\,]$. Define a global deformation of $\mathfrak{g}$ over the affine line $A = \mathrm{Spec}(\mathbb{C}[t])$ by $(\mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t], [\,,\,]_t)$ with the bracket $[\,,\,]_t = (t-1)[\,,\,]$. This is a global deformation of $\mathfrak{g}$ which is non-trivial in the sense that there are two fibers which are non-isomorphic as $\mathbb{C}$-Lie algebras (namely the fibers above $t = 0$ and $t = 1$; the Lie algebra above $t = 1$ is abelian). But admittedly, it is trivial in some sense.

(2) Let $\pi : C \to S$ be a smooth family of curves with marked points $\sigma_1, \ldots, \sigma_n : S \to C$. We explained in Sections 2 and 3 how to associate to $\pi : C \to S$ a global deformation $\mathfrak{g}_A$ of Lie algebras of the Lie algebra $\mathfrak{g}$ of regular vector fields on $C_0 \setminus (\sigma_1(0) \cup \cdots \cup \sigma_n(0))$, with $\mathrm{Spec}(A) = S$. In case $\mathfrak{g}_A$ is a not a free A-module, this deformation cannot be trivial (as a global deformation of Lie algebras).

(3) This is the example of Fialowski–Schlichenmaier [8]. Elliptic curves can be parametrized in the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ by e_1, e_2 and e_3 such that the curve is given by the equation

$$Y^2 Z = 4(X - e_1 Z)(X - e_2 Z)(X - e_3 Z)$$

with $e_1 + e_2 + e_3 = 0$ and $\triangle = 16(e_1 - e_2)^2 (e_1 - e_3)^2 (e_2 - e_3)^2 \neq 0$. $\triangle \neq 0$ assures that the curve is non-singular. These equations form a family of curves over $B := \{(e_1, e_2, e_3) \mid e_1 + e_2 + e_3 = 0, e_i \neq e_j \ \forall i \neq j\}$. Completing the parameter space B to $\hat{B} := \{(e_1, e_2, e_3) \mid e_1 + e_2 + e_3 = 0\}$ admits singular cubics. Partial degeneration (e.g. $e_1 = e_2 \neq e_3$) leads to the *nodal cubic*

$$E_N : Y^2 Z = 4(X - eZ)^2(X + 2eZ),$$

while overall degeneration (e.g. $e_1 = e_2 = e_3$) leads to the *cuspidal cubic*

$$E_C : Y^2 Z = 4X^3.$$

The nodal cubic is singular, but still stable and constitutes therefore a point of the boundary in the Deligne–Mumford compactification of the moduli space, while the cuspidal cubic is not. As the genus of the desingularization (i.e. normalization) decreases, the desingularization of E_N and E_C is the projective line $\mathbb{P}^1_{\mathbb{C}}$. Fialowski and Schlichenmaier extract from the above family of curves a family over $\mathbb{C}[t]$ and compute explicitly the relations of the Lie algebra of vector fields on the family of elliptic curves. It turns out that the generic fiber Lie algebra is not isomorphic to the Witt algebra, while the fiber at $t = 0$ is.

The point which makes this example less trivial than the preceding one is that in any neighborhood of 0, the restriction of the global deformation to this neighborhood remains non-trivial, while in the first two examples, there are neighborhoods where the deformation is trivial.

Fialowski and Schlichenmaier pointed out in [8] that this is an example of an infinitesimally rigid Lie algebra (because the Witt algebra has trivial second cohomology space with adjoint coefficients) which has non-trivial global deformations.

The maps I and N permit one in principle to define lots of examples of this third type. The hard part is to compute the relations of the so-defined Lie algebras in order to render the examples explicit.

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References

[1] A. Altman, S. Kleiman, Introduction to Grothendieck Duality Theory, in: Lecture Notes in Mathematics, vol. 146, Springer Verlag, Heidelberg, 1970.
[2] Ichiro Amemiya, Lie algebra of vector fields and complex structures, J. Math. Soc. Japan 27 (4) (1975) 545–549.
[3] A. Beilinson, B.L. Feigin, B. Mazur, Algebraic conformal field theory, preprint.
[4] Steven Diaz, Joe Harris, Ideals of singular plane curves, Trans. Amer. Math. Soc. 309 (2) (1988) 433–468.
[5] Dan Edidin, Notes on the Construction of the Moduli Space of Curves. Recent Progress in Intersection Theory (Bologna, 1997), in: Trends Math., Birkhäuser Boston, Boston, MA, 2000, pp. 85–113.
[6] David Eisenbud, Commutative Algebra with a View Towards Algebraic Geometry, in: Springer Graduate Texts in Mathematics, vol. 150, Springer Verlag, New York, 1995.
[7] Fialowski Alice, Fuchs Dimitry, Construction of Miniversal Deformations of Lie Algebras, J. Funct. Anal. 161 (1999) 76–110.
[8] Alice Fialowski, Martin Schlichenmaier, Global deformations of the Witt algebra of Krichever–Novikov type, Commun. Contemp. Math. 5 (6) (2003) 921–945.
[9] Concettina Galati, Number of moduli of irreducible families of plane curves with nodes and cusps, preprint on the arXiv:0704.0618.
[10] Janusz Grabowski, Isomorphisms and ideals of the Lie algebra of vector fields, Invent. Math. 50 (1978) 13–33.
[11] Jean Giraud, Cohomologie non abélienne. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 179, Springer Verlag, Heidelberg, 1971.
[12] Grothendieck Alexander, et al., SGA 1: Revêtements étales et Groupe Fondamental, in: Lecture Notes in Mathematics, vol. 224, Springer Verlag, Heidelberg, 1971.
[13] Joe Harris, Ian Morrison, Moduli of Curves, in: Springer GTM, vol. 187, 1998.
[14] Robin Hartshorne, Algebraic Geometry, in: Springer Graduate Texts in Mathematics, vol. 52, 1977.
[15] Herwig Hauser, Gerd Müller, Affine varieties and Lie algebras of vector fields, Manuscripta Math. 80 (1993) 309–337.
[16] Finn F. Knudson, The projectivity of the moduli space of stable curves II: The stacks $M_{g,n}$, Math. Scand. 52 (1983) 161–199.
[17] Gérard Laumon, Laurent Moret-Bailly, Champs algébriques, in: Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 39, Springer Verlag, Berlin, 2000.
[18] David Metzler, Differentiable stacks, math.DG/0306176.

[19] Martin Schlichenmaier, Oleg Sheinman, Knizhnik–Zalmolodchikov equations for positive genus and Krichever–Novikov algebras, Russian Math. Surveys 59 (4) (2004) 737–770.

[20] Thomas Siebert, Lie algebras of derivations and affine algebraic geometry over fields of characteristic 0, Math. Ann. 305 (1996) 271–286.

[21] Angelo Vistoli, Notes on Grothendieck topologies, fibered categories and descent theory, in: Barbara Fantechi, et al. (Eds.), Fundamental Algebraic Geometry: Grothendieck's FGA Explained, Amer. Math. Soc., 2005.